Relational Variables in Graph Theory

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Abstract

We systematically consider simple relational variables – relative variables, ratio variables and dilatational variables – for Graph Theory. We apply these to simplifying graph inequalities and elucidating a large number Graph-Theoretically significant probability-valued variables. This material has further use in developing network stucture quantifiers. It represents interaction between Similarity Geometry, and basic Shape Theory in the sense of Kendall, with Graph Theory.

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1 Introduction

We apply a simple version of Relational Theory [44, 14, 16, 17, 19, 20, 21, 25, 27, 28, 30, 33, 32, 36, 37, 38, 39, 40, 45] to Graph Theory [4, 5, 8, 15, 22, 26, 23, 34, 24, 42, 31]. 'Simple' here refers to ratio variables (Sec 2), relative variables (Sec 3) and dilatational variables (Sec 4). These corresponds to quotienting out, respectively, dilations, translations, and both at once. More general relational theory quotients out larger geometrical isomorphism groups [16, 19, 25, 35, 36]; its scale-free case is Shape Theory [16, 20, 25, 36, 38, 39, 40, 45]. Another descriptor for the dilatational case is Kendall's preshape theory.

Using ratios is useful in comparing graphs of different sizes. This is the general rationale behind the current article including rational fractions as well as the more habitually used cardinal functions. We carry this out in Part I for pre-topological consideration of graphs and in Part II for topological ones.

We apply this approach to reformulating simple graph-theoretic inequalities. This parallels our previously tidying up some geometrical inequalities in this way [40, 41]. We also form probability-valued graph quantifiers in places of cardinal-valued ones. This article's considerations are subsequently useful in further developing the theory of structure of networks: a very relevant modern topic [29, 31].

2 Theory of ratios

Structure 0 Suppose we are given independent scalar quantities

$$q_I \ , \ \ I \ = \ 1 \ \text{ to } \ Q \ . \tag{1}$$

Then $R \ := \ Q \ - \ 1$ independent ratios are supported.

Structure 1 One presentation for these is

$$\mathcal{R}_i \ := \ \frac{q_i}{q_Q} \ , \ \ i \ = \ 1 \ \text{ to } \ R \ : \tag{2}$$

simple ratios.

Structure 2 Another is

$$\mathcal{N}_i \ := \ \frac{q_i}{\sqrt{\sum_{i=0}^{Q} {q_i}^2}} \ : \tag{3}$$

normalized ratios. These obey

$$\sum_{i=0}^{Q} {\mathcal{N}_i}^2 \ = \ 1 \ . \tag{4}$$

Example 1 *Stereographic coordinates* on $\mathbb{S}^n$ are simple ratios.

Example 2 *Inhomogeneous coordinates* in projective spaces are simple.

Example 3 $\sin\theta$ and $\cos\theta$ are normalized ratios on $\mathbb{S}^1$.

Example 4 *Unit normal components* are normalized ratios, generalizing Example 3 from $\mathbb{S}^1$ to $\mathbb{S}^n$.

Structure 3

$$\mathcal{P}_i \ = \ \frac{q_i}{\sum_{i=0}^{Q} q_i} \ . \tag{5}$$

These obey

$$\sum_{i=0}^{n} \mathcal{P}_i \ = \ 1 \ . \tag{6}$$

There is an issue of sum of mods versus sum if the quantities can be negative.

Example 5 *Partial pressures* are of this nature. So is the dimensionless version of partial moments of inertia.

Structure 4 Extend to pth moments, and pth roots thereof.

A more general possibility is

$$\mathcal{R}_i^{(p)} \ = \ \frac{q_i}{(\sum_{i=0}^{p} q_i)^{\frac{1}{p}}} \tag{7}$$

It allows for L^p norms; $\mathcal{P}_i$ and $\mathcal{N}_i$ are the $p \ = \ 1, 2$ cases respectively.

The above account is not exhaustive but does cover many practically encountered ones.

If the q_i are dimensionful – all matchingly – the ratios formed from them are dimensionless. This comes with extra protections.

Example 1 Physical quantities to dimensionless groups.

Structure 1 *Ratiospace*

$$\mathfrak{R}\text{atio}(N) \tag{8}$$

is the space of ratios of N-tuples of scalar quantities. This is an abstract copy of the spatially 1-d version of Kendall's preshape space. It is topologically and metrically $\mathbb{RP}^n$ or $\mathbb{S}^n$ [16, 38, 46], depending on whether mirror images are or are not identified. The above examples can then furthemore be viewed as coordinates on ratiospace.

Remark 3 Physical treatments often follow €xample 1's lines but, as far as I know, have never been considered from the $\mathfrak{R}$atio point of view. The $\mathfrak{R}$atio point of view requires coordinate singularities to be explicitly pointed out, atlases declared, and minimal atlases (to cover) alongside maximal atlases (extent of compatibility).

3 Theory of relative quantities

Definition 1 For equidimensional quantities q_I taken to belong to the same space in possession of an addition operation, the *difference variables* are

$$r_{IJ} := q_I - q_J . \tag{9}$$

Structure 1 Let us next introduce maximal and minimal quantities q_{max}, q_{min} within a set of equidimensional scalar quantities. We can then define the following.

Definition 2 The *upper-shifted relative quantity*

$$q_u := q_{\text{max}} - q \tag{10}$$

Definition 3 The *lower-shifted relative quantity*

$$q_l := q - q_{\text{min}} . \tag{11}$$

Definition 4 The *spread*

$$\Delta q := q_{\text{max}} - q_{\text{min}} . \tag{12}$$

Remark 1 The above quantities obey the following dependency.

$$q_u + q_l = q_{\text{max}} - q + q - q_{\text{min}} = q_{\text{max}} - q_{\text{min}} = \Delta q . \tag{13}$$

Structure 2 Let us next introduce

$$S := \sum_I q_I : \tag{14}$$

the *sum* of all the quantities in an equidimensional set. We can then furthermore define the following.

Definition 5 The *average*,

$$\langle q \rangle := \frac{\sum_I q_I}{N} . \tag{15}$$

Remark 2 Sum is also *first total moment*.

Structure 2 Not all of the r_{IJ} are independent. We can however choose an independent subset of them. Under certain circumstances, such as the presence of $\sum q_I^2$ (or $\sum m_I q_I^2$: weighted second moment), we are furthermore interesting in *diagonalizing linear combinations* R_i of the q_I. i here runs over 1 index value less than I: from 1 to $R = Q - 1$. When the q_I are positions on flat space, the R_i are *Jacobi coordinates.*

We can also now consider further relative quantities such as

$$q_I - \sum_I q_I \tag{16}$$

We can just as well introduce nth total moments

$$M_n := \sum_I q_I^n : \tag{17}$$

and their pth roots, among which the second is the root mean sum (r.m.s.).

The *r.m.s. fluctuation*

$$q_I - \sqrt{\sum_I {q_I}^2} . \tag{18}$$

The *fluctuation*

$$q_I - \langle q_I \rangle \tag{19}$$

4 Theory of dilatational quantities

Combining the previous two subsections, we can form such as the following.

1) The dilatational quantities

$$Q_{IJKL} := \frac{q_I - q_J}{q_K - q_L} . \quad (20)$$

2) The diagonalized relative quantities' ratios

$$R_{ij} := \frac{R_i}{R_j} . \quad (21)$$

3) Normalized shifted variables

$$n_{\mathrm{u}} := \frac{q_{\mathrm{max}} - q}{q} , \quad n_{\mathrm{l}} := \frac{q - q_{\mathrm{min}}}{q} . \quad (22)$$

4) A *normalized* notion of *fluctuation*,

$$f_n := \frac{q_I - \langle q \rangle}{\langle q \rangle} = 1 - \hat{q}_i . \quad (23)$$

5) A *normalized* notion of *spread*,

$$q_{\mathrm{c}} = \frac{\Delta\rho}{\rho} . \quad (24)$$

Structure 1 Suppose q runs from A to B, both finite. Then

$$q' := \frac{q - A}{B - A} \quad (25)$$

runs from 0 to 1. This is one way of obtaining standardized-finite-range – unit-interval – and probability-valued quantities.

Structure 2 Another is applying the mod sign to quantities running from -1 to 1.

Remark 1 There are of course many others, but these are the two used in the current Article. If one or both of A and B were not finite, compactification can be attained using certain functions, for instance tan.

Part I
Pre-topological Graph Theory

5 Graphs

5.1 Standard presentation

Definition 1 Let

$$\mathfrak{G} \;=\; (\mathfrak{V}, \mathfrak{E}) \tag{26}$$

be a *graph* [22, 26, 24, 31]: a collection of vertices v, w... forming the *vertex set* $\mathfrak{V}$ some of which are joined by edges $e \;=\; v\,w$ forming the *edge set* $\mathfrak{E}$. It is of *order*

$$V \;:=\; |\,\mathfrak{V}\,| \;=\; |\mathfrak{G}| \;=\; N \tag{27}$$

and *size*

$$E \;:=\; |\,\mathfrak{E}\,|\,. \tag{28}$$

Remark 1 We restrict ourselves to one edge per distinct-vertex pair.[1] Then for each N, the number of possible edges runs from 0 to

$$E_{\mathrm{max}} \;=\; \binom{N}{2} \;=\; \frac{N\,(N\;-\;1)}{2}\;. \tag{29}$$

Structure 1 The *complement* of a graph is the graph obtained by switching edges and non-edges between each vertex pair. Most graphs occur in complementary pairs, though a few are self-complementary.

Lemma 1 A necessary but not sufficient condition for self-complementarity is

$$E \;=\; E_{\mathrm{crit}} \;=\; \frac{E_{\mathrm{max}}}{2} \;=\; \frac{N\,(N\;-\;1)}{4}\;. \tag{30}$$

Definition 3 A *[graph]* [49] is a graph modulo complementation.

Remark 4 Our starting package for building relative, relational and dilatational quantities is thus $V \;=\; N$, E, and two particular fixed values thereof, E_{max} and E_{crit}. We deal with graphs first, eventually covering also [graphs], and then moving on to the spaces formed by each of these, $\mathfrak{G}$raph and [$\mathfrak{G}$raph] (Sec 9).

5.2 Ratio variables

Remark 1 Our starting package thus supports one ratio variable as follows.

Definition 1 The *size to order ratio*

$$\eta(\mathfrak{G}) \;:=\; \frac{E(\mathfrak{G})}{|\mathfrak{G}|} \;=\; \frac{E}{N}\;. \tag{31}$$

Remark 2 The last form here is used whenever the graph being referred to is clear. In subsequently introducing objects we skip making reference to the particular graph as a default.

Remark 3 Our starting package also supports two particular normalizations, as follows.

Definition 2 *Edge density*

$$\rho \;:=\; \frac{E}{E_{\mathrm{max}}}\;. \tag{32}$$

Remark 4 This runs from 0 to 1: standardized-range alias probability-valued.

Definition 3 *Ramsey density*

$$\sigma \;:=\; \frac{E}{E_{\mathrm{crit}}}\;. \tag{33}$$

Remark 5 This is numerically edge bidensity,

$$\sigma \;=\; 2\,\rho \tag{34}$$

running from 0 to 2.

[1] I.e. the *simple graphs*, as opposed to multigraphs and/or looped graphs [31].

5.3 Relative variables

Remark 1 One relative variable is also supported, as follows.

Definition 1 The 1-*d Euler characteristic*

$$\chi \ := \ V \ - \ E \ = \ N \ - \ E \ . \tag{35}$$

Remark 2 So is one non-relative total variable, as follows.

Definition 2 The *sum of qualitative cases* [38]

$$\Sigma \ := \ V \ + \ E \ = \ E \ + \ E \ . \tag{36}$$

Remark 3 Aside from Definition 1 being relatively significant, it is topologically-significant, extending in particular to

$$\chi \ := \ V \ - \ E \ + \ F \tag{37}$$

in cases in which face count is meaningful.

Remark 4 For fixed N, χ runs from 0 to

$$\frac{N\,(\,1 \ + \ N\,)}{2} \ . \tag{38}$$

Σ runs from N to

$$\frac{N\,(\,N \ + \ 1\,)}{2} \ . \tag{39}$$

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Remark 5 Graph edge counts support two shifted variables of note, as follows.

Definition 3 *Maximal edge discrepancy*

$$MED \ := \ E_{\max} \ - \ E \ . \tag{40}$$

Definition 4 *Critical edge discrepancy*

$$CED \ := \ E_{\mathrm{crit}} \ - \ E \ . \tag{41}$$

5.4 Dilatational variables

The following further dependent ratios can then be formed.

Definition 1 The *Euler ratio*

$$\mathcal{E} \ := \ \frac{\chi}{\Sigma} \ = \ \frac{V \ - \ E}{V \ + \ E} \ = \ \frac{1 \ - \ \eta}{1 \ + \ \eta} \ . \tag{42}$$

Remark 2 This runs over the range

$$\left(\frac{1 \ - \ N}{1 \ + \ N} \ , \ \ 1 \right) \ . \tag{43}$$

To standardize range to (0, 1), we can shift and rescale to

$$P_{\mathrm{Euler}} \ := \ (\,1 \ + \ N^{-1}\,)(\,1 \ + \ \eta^{-1}\,)^{-1} \ . \tag{44}$$

Definition 2 The *complementary edge density*

$$\mathcal{M} \ := \ \frac{MED}{E_{\max}} \ = \ \frac{E_{\max} \ - \ E}{E_{\max}} \ = \ 1 \ - \ \rho \ . \tag{45}$$

Remark 3 This runs from 0 to 1.

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Definition 3 The *relative* alias *Ramsey* fraction is

$$\mathcal{R} \ = \ \frac{CED}{E_{\mathrm{crit}}} \ := \ \frac{E_{\mathrm{crit}} \ - \ E}{E_{\mathrm{crit}}} \ = \ 1 \ - \ \sigma \ = \ 1 \ - \ 2\,\rho \ . \tag{46}$$

Remark 4 This runs from -1 to 1.

Remark 5 For [graphs], the key quantity is E_{crit}, supporting the ratio σ and the following.

Definition 4 The *unsigned Ramsey fraction*,

$$\mathcal{U} \;=\; \left| \frac{E_{\max} \;-\; E}{E_{\text{crit}}} \right| \;=\; |\,1 \;-\; \sigma\,| \;=\; |\,1 \;-\; 2\,\rho\,|\;. \tag{47}$$

Remark 6 This is a probability-valued function of note in the [graph] setting [49].

6 Degrees

6.1 Standard presentation

Definition 1 The *degree* $d(w)$ alias *valency* v_w of a vertex v is the number of edges adjacent to it.

Definition 2 A graph's *minimal degree* is

$$\delta \ := \ \begin{matrix}\min \\ w \, \in \, \mathfrak{V}(\mathfrak{G})\end{matrix} \ d(w) \ . \tag{48}$$

Its *maximal degree* is

$$\Delta \ := \ \begin{matrix}\max \\ w \, \in \, \mathfrak{V}(\mathfrak{G})\end{matrix} \ d(w) \ . \tag{49}$$

6.2 Degree ratios

Definition 1 The *fractional degree within a* graph is

$$f(v) \ = \ \frac{d(v)}{\Delta} \ . \tag{50}$$

Definition 2 The *minimax degree ratio* of a given graph is

$$\mathcal{D} \ = \ \frac{\delta}{\Delta} \ . \tag{51}$$

Remark 1 This is 1 iff the graph is nontrivial and regular.

Definition 3 The *degree per unit size*

$$u \ := \ \frac{d(v)}{N} \ . \tag{52}$$

6.3 Relative degrees

Definition 1 The *degree spread* of a fixed graph is

$$\vartriangle \ = \ \Delta \ - \ \delta \ . \tag{53}$$

Remark 1 Let us also introduce the following non-relative quantity.

Definition 2

$$\Sigma \ = \ \Delta \ + \ \delta \ . \tag{54}$$

Remark 2 This is quite often scaled by $\frac{1}{2}$ so as to pick out the centre of the spread.

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Remark 3 Two shifted degrees of note are as follows.

Definition 3 The *lower shifted degree*

$$d_{\mathrm{l}}(v) \ := \ d(v) \ - \ \delta \ . \tag{55}$$

Definition 4 The *upper shifted degree*

$$d_{\mathrm{u}}(v) \ := \ \Delta \ - \ d(v) \ . \tag{56}$$

Remark 4 There is of course one dependency between the above relative quantities,

$$d_{\mathrm{l}} \ + \ d_u(\Delta \ - \ d) \ + \ (d \ - \ \delta) \ = \ \Delta \ - \ d \ + \ d \ - \ \delta \ = \ \Delta \ - \ \delta \ = \ \vartriangle \ . \tag{57}$$

By this, we only make use of one of the shifted degrees.

Remark 5 A further non-relative quantity of note is as follows.

Definition 5

$$\langle d \rangle \ := \ \frac{1}{|\mathfrak{G}|} \sum_{v \, \in \, \mathfrak{G}} d(v) \ = \ \frac{1}{N} \sum_{v \, \in \, \mathfrak{G}} d(v) \ . \tag{58}$$

Definition 6 The '*unnormalized fluctuation*'

$$fluc(v) \ := \ d(v) \ - \ \langle d \rangle \ . \tag{59}$$

6.4 Dilatational notions of degree

Definition 1 The '*average-normalized degree*'

$$\hat{d} \;=\; \frac{d}{\langle d \rangle}\;. \tag{60}$$

We can next form the following.

Definition 2 The *degree increment*

$$d_{\mathrm{I}} \;:=\; \frac{\scriptstyle\Delta}{\Delta} \;=\; \frac{(\,\Delta\;-\;\delta\,)}{\Delta} \;=\; 1\;-\;\mathcal{D}\;. \tag{61}$$

Definition 3 The *half-normalized spread*

$$HNS \;:=\; \frac{\scriptstyle\Delta}{\scriptstyle\Sigma} \;=\; \frac{\Delta\;-\;\delta}{\Delta\;+\;\delta} \;=\; \frac{1\;-\;\mathcal{D}}{1\;+\;\mathcal{D}}\;. \tag{62}$$

Definition 4

$$\frac{\Delta\;-\;d(v)}{\Delta} \;=\; 1\;-\;f(v)\;. \tag{63}$$

Definition 6 The *normalized degree fluctuation*

$$f_{\mathrm{n}} \;:=\; \frac{(\langle d \rangle\;-\;d)}{\langle d \rangle} \;=\; 1\;-\;\hat{d}\;. \tag{64}$$

6.5 Applications

Remark 1 To run from 0 to 1 and thus be probability-valued, one would use instead the *shifted degree fluctuation*

$$\frac{\langle d \rangle\;-\;d\;-\;\delta}{\scriptstyle\Delta}\;. \tag{65}$$

Remark 2 *Ore's sufficiency condition* [10, 23, 26] *for Hamiltonianness* [26]

$$d(v)\;+\;d(w)\;>\;|\mathfrak{G}|\;=\;N \tag{66}$$

becomes, in terms of ratio quantities,

$$u(v)\;+\;u(w)\;>\;1\;. \tag{67}$$

Remark 3

$$d(v)\;>\;\frac{|\mathfrak{G}|}{2}\;=\;\frac{N}{2} \tag{68}$$

– a simplified subcase priorly used by G. Dirac [7] – becomes

$$u(v)\;>\;\frac{1}{2}\;. \tag{69}$$

Limitation 1 δ and Δ are however already compared with the ratio η in a Corollary of Euclid's Degree-sum Theorem. I.e.

$$\delta\;\leq\;\lfloor 2\,\eta \rfloor \tag{70}$$

and

$$\Delta\;\leq\;\lceil 2\,\eta \rceil\;. \tag{71}$$

This illustrates that there can be limitations on conceptualizing in terms of ratios when the initial quantities are already dimensionless.

7 Graph comparers and subgraphs

Definition 1 A subgraph $\mathfrak{H}$ of $\mathfrak{G}$ is a subset of $\mathfrak{G}$'s vertices alongside a subset of $\mathfrak{G}$'s edges.

7.1 Relative quantities

Definition 1 The *relative order*

$$V(\mathfrak{G} - \mathfrak{H}) = |\mathfrak{G}| - |\mathfrak{H}| . \tag{72}$$

Definition 2 The *relative size*

$$E(\mathfrak{G} - \mathfrak{H}) = E(\mathfrak{G}) - E(\mathfrak{H}) . \tag{73}$$

Remark 1 In each case add mod bars for the unsigned version, which is then symmetric between two not necessarily nested graphs.

7.2 Ratio quantities

Definition 1 The *order ratio*

$$V(\mathfrak{H}|\mathfrak{G}) = \frac{|\mathfrak{H}|}{|\mathfrak{G}|} . \tag{74}$$

Definition 2 The *relative size*

$$E(\mathfrak{H}|\mathfrak{G}) = \frac{E(\mathfrak{H})}{E(\mathfrak{G})} . \tag{75}$$

Remark 1 Here our own notation $(\mathfrak{H}|\mathfrak{G})$ reads 'of $\mathfrak{H}$ within $\mathfrak{G}$.

Definition 3 The *symmetric order ratio* of two graphs $\mathfrak{G}$, $\mathfrak{H}$ is

$$V_{\mathrm{S}}(\mathfrak{H}|\mathfrak{G}) := \frac{1}{2}\left(\frac{|\mathfrak{H}|}{|\mathfrak{G}|} + \frac{|\mathfrak{H}|}{|\mathfrak{G}|}\right) . \tag{76}$$

Remark 2 The factor of 1/2 ensures it returns 1 when $\mathfrak{G} = \mathfrak{H}$.

Structure 1 The *symmetric order discrepancy* of two graphs $\mathfrak{G}$, $\mathfrak{H}$ is

$$V_{\mathrm{D}}(\mathfrak{H}|\mathfrak{G}) := V_{\mathrm{S}}(\mathfrak{H}|\mathfrak{G}) - 1 = \frac{1}{2}\left(\sqrt{\frac{|\mathfrak{H}|}{|\mathfrak{G}|}} - \sqrt{\frac{|\mathfrak{H}|}{|\mathfrak{G}|}}\right)^2 = \frac{1}{2}\left(\frac{(|\mathfrak{H}| - |\mathfrak{G}|)^2}{|\mathfrak{H}|\,|\mathfrak{G}|}\right)^2 . \tag{77}$$

Remark 3 Symmetric size ratio and discrepancy are defined similarly.

8 Notions of distance on graphs

Definition 0 The *length* $l(\mathrm{P})$ *of a path* P (including when realized as a subgraph) is the number of edges that it contains. The *length* $l(\mathrm{C})$ *of a cycle* C is defined likewise.

Definition 1 The *distance* between vertices x and y in a graph $\mathfrak{G}$ [26] is given by the following.

$$\mathrm{d}(x,y) \;=\; \min_{\text{paths P} \;:\; x \;\longrightarrow\; y} (l(\mathrm{P})) \tag{78}$$

(For x and y not path-connected, $\mathrm{d}(x,y) \;:=\; \infty$.)

Definition 2 The *eccentricity* $e(v)$ of a vertex v [26] is

$$ecc(v) \;:=\; \max_{w \;\in\; \mathfrak{V}(\mathfrak{G})} (d(v,w)) \;. \tag{79}$$

Remark 1 Concatenating (78), (79),

$$ecc(v) \;=\; \max_{w \;\in\; \mathfrak{V}(\mathfrak{G})} \left(\min_{\text{paths P} \;:\; x \;\longrightarrow\; y} (l(\mathrm{P})) \right) \;. \tag{80}$$

Definition 3 The *radius* $r(\mathfrak{G})$ of a graph $\mathfrak{G}$ [26] is

$$rad(\mathfrak{G}) \;:=\; \min_{v \;\in\; \mathfrak{V}(\mathfrak{G})} (ecc(v)) \;. \tag{81}$$

Remark 1 Concatenating (79), (81),

$$rad(\mathfrak{G}) \;=\; \min_{v \;\in\; \mathfrak{V}(\mathfrak{G})} \left(\max_{w \;\in\; \mathfrak{V}(\mathfrak{G})} \left(\min_{\text{paths P} \;:\; x \;\longrightarrow\; y} (l(\mathrm{P})) \right) \right) \;. \tag{82}$$

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Definition 4 A vertex v is *central* if

$$ecc(v) \;=\; rad(\mathfrak{G}) \;. \tag{83}$$

The *centre* of $\mathfrak{G}$ is the set of central vertices

$$\mathfrak{C}(\mathfrak{G}) \;:=\; \{\, v \;\in\; \mathfrak{V}(\mathfrak{G}) \,|\, e(v) \;=\; r(\mathfrak{G}) \,\} \;. \tag{84}$$

Definition 5 The *diameter* $diam(\mathfrak{G})$ of a graph $\mathfrak{G}$ [26] is

$$diam(\mathfrak{G}) \;:=\; \max_{v \;\in\; \mathfrak{V}(\mathfrak{G})} (ecc(v)) \tag{85}$$

Remark 1 Concatenating (79) and (85),

$$diam(\mathfrak{G}) \;=\; \max_{w,v \;\in\; \mathfrak{V}(\mathfrak{G})} \left(\min_{\text{paths P} \;:\; x \;\longrightarrow\; y} (l(\mathrm{P})) \right) \;. \tag{86}$$

Definition 6 A vertex v is *peripheral* if

$$ecc(v) \;=\; diam(\mathfrak{G}) \;. \tag{87}$$

Remark 1 These obey

$$rad \;\leq\; diam \;\leq\; 2\,rad \;. \tag{88}$$

8.1 Distance ratios

Definition 1 The *diameter per radius*

$$\mathcal{Z} \;:=\; \frac{diam}{rad} \;. \tag{89}$$

Definition 2 The *acentricity*

$$\overline{\mathcal{C}} \;:=\; \frac{rad(\mathfrak{G})}{e(v)} \;. \tag{90}$$

Definition 3 The *aperipherality*

$$\overline{\mathcal{P}} \ := \ \frac{ecc(v)}{rad(\mathfrak{G})} \ . \tag{91}$$

Remark 1 These are both probability-valued, taking value 1 in the central and peripheral cases respectively.

Remark 2 The above two names are, as far as I am aware, new to the current article.

Remark 3 On the other hand,

$$1 \ \leq \ \mathcal{Z} \ \leq \ 2 \tag{92}$$

follows from (88)

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Structure 1 If we consider the *diameter-to-radius increment*

$$\zeta \ := \ \frac{diam \ - \ rad}{rad} \ = \ \mathcal{Z} \ - \ 1 \ , \tag{93}$$

however, we arrive at a further probability-valued variable.

Remark 3 Finally, a standard definition we make use of later on is as follows.

8.2 Cycle quantifiers

Definition 1 The *cycle length* $\mathrm{l}(c)$ of a cycle c in a graph $\mathfrak{G}$ is the number of edges in the cycle.

Definition 2 The *girth* g of a graph $\mathfrak{G}$ is

$$g \ := \ \min_{\text{cycles } c \ \in \ \mathfrak{G}} \mathrm{l}(c) \ . \tag{94}$$

Definition 2 The *circumference* C of a graph $\mathfrak{G}$ is

$$C \ := \ \max_{\text{cycles } S \ \in \ \mathfrak{G}} \mathrm{l}(S) \ . \tag{95}$$

Lemma 1 The inequality

$$3 \ \leq \ g \ \leq \ l \ \leq \ C \ \leq \tag{96}$$

holds trivially.

Definition 3 The *cycle range*

$$\triangle c \ = \ C \ - \ g \ . \tag{97}$$

Definition 4 The *standardized cycle fraction*

$$\sigma \ := \ \frac{E}{\Delta C} \ = \ \frac{1 \ - \ g}{C \ - \ g} \ . \tag{98}$$

Remark 1 This is only defined for $C \ > \ g$.

Definition 5 *Circumference density*

$$c \ := \ \frac{C}{|G|} \ . \tag{99}$$

Definition 6 *Girth fraction of circumference*

$$\mathcal{C} \ := \ \frac{g}{C} \ . \tag{100}$$

Corollary 1 σ is probability-valued and

$$c \ \leq \ 1 \ . \tag{101}$$

$\mathcal{C} \ = \ 1$ is Hamiltonian.

Corollary 2

$$\overline{\mathcal{H}} \ := \ 1 \ - \ \mathcal{C} \ = \ \frac{\Delta c}{C} \ : \tag{102}$$

– cycle range fraction of circumference – is an anHamiltonianness, i.e. measure of departure from Hamiltonianness.

Proposition 1

$$\mathcal{I} \ := \ \frac{1 \ - \ \mathcal{J}}{3 \ - \ \mathcal{C}} \tag{103}$$

is probability-valued.

Remark 1 This motivates the following.

Definition 7 The *circumference per unit cycle length*

$$\mathcal{J} \ := \ \frac{c}{l(S)} \tag{104}$$

9 Spaces of graphs relationally characterized

9.1 Basic notions

Definition 1 The *space of graphs of fixed order N*

$$\mathfrak{G}\text{raph}(N) \ , \tag{105}$$

and the *space of graphs up to fixed order N*,

$$\mathfrak{G}\text{raph}[N] \ = \ \coprod_{n=0}^{N} \mathfrak{G}\text{raph}(n) \ . \tag{106}$$

Remark 1 The latter is furthermore a first approximation to the *space of finite graphs*

$$\mathfrak{G}\text{raph} \ = \ \coprod_{n \, \in \, \mathbb{N}_0} \mathfrak{G}\text{raph}(n) \ . \tag{107}$$

Remark 2 This triplication can be stated for each (sub)space of graphs in this Article; we shall not bother to do so again.

Structure 2 Let us denote the *space of self-complementary graphs* by

$$\mathfrak{SC} \ . \tag{108}$$

Remark 3 Let us now jointly centre the most symmetric cases (at present in the sense of self-complementarity). Unlike in [49], we use d_{R} as dependent variable: a relative-and-rational alias dilatational view of $\mathfrak{G}\text{raph}(N)$.

9.2 $\mathfrak{G}$raph ratios

Definition 1 The *fractional degree within* $\mathfrak{G}\text{raph}(N)$ is

$$F(v) \ = \ \frac{d(v)}{\Delta(\mathfrak{G}\text{raph})} \ = \ \frac{d(v)}{N \ - \ 1} \ . \tag{109}$$

Remark 2 The maximal value of degree over all simple graphs of order N is

$$d_{\text{max}} \ = \ N \ - \ 1 \tag{110}$$

and its minimal value is 0. This gives a further bunch of normalized quantities:

$$\frac{d}{d_{\text{max}}} \ , \ \ \frac{\delta}{d_{\text{max}}} \ , \ \ \frac{\Delta}{d_{\text{max}}} \ , \ \ \frac{\vartriangle}{d_{\text{max}}} \ , \ \ \frac{\langle d \rangle}{d_{\text{max}}} \ ...$$

This subsection's remaining ratios are of the conceptual type

$$\frac{|\text{Subspace of graphs}|}{|\text{Space of graphs}|} \ . \tag{111}$$

Example 1) The *connected fraction* is

$$c(N) \ = \ \frac{|\mathfrak{C}\mathfrak{G}\text{raph}(N)|}{|\mathfrak{G}\text{raph}(N)|} \ . \tag{112}$$

Example 2) The *forest fraction* is

$$f(N) \ = \ \frac{|\mathfrak{F}\text{orest}(N)|}{|\mathfrak{G}\text{raph}(N)|} \ . \tag{113}$$

Remark 3 One can also consider the fraction exhibiting k cycles, $c^k(N)$ (with $c^0(N) \ = \ t(N)$).

Remark 2 These $t^k(N)$ are partial counts:

$$\sum_k t^k(N) \ = \ 1 \ . \tag{114}$$

Example 3) The *tree fraction* is

$$t(N) \ = \ \frac{|\mathfrak{T}\text{ree}(N)|}{|\mathfrak{G}\text{raph}(N)|} \ . \tag{115}$$

9.3 $[\mathfrak{G}\mathbf{raph}](N)$

Definition 1 Let

$$[\mathfrak{G}\text{raph}] \tag{116}$$

be used to denote the space of [graphs].

Remark 1 Unlike in [49], we use $u_{\rm R}$ as dependent variable: a relative-and-rational alias dilatational view of $[\mathfrak{G}\text{raph}](N)$.

9.4 [Graph] ratios

Example 1) The *[graph] fraction*

$$[g](N) \ := \ \frac{|[\mathfrak{G}\text{raph}](N)|}{|\mathfrak{G}\text{raph}(N)|} \ . \tag{117}$$

The *self-complementary fraction*

$$s(N) \ = \ \frac{|\mathfrak{S}\mathfrak{C}(N)|}{|\mathfrak{G}\text{raph}(N)|} \ . \tag{118}$$

For basic reasons given in e.g. [49] these two obey

$$2\,[g](N) \ + \ s(N) \ = \ 1 \ . \tag{119}$$

For $N \ = \ 2$ or $3(\text{mod}4)$, this simplifies to

$$s(N) \ = \ 0 \ , \ \ [g](N) \ = \ \frac{1}{2} \ . \tag{120}$$

Example 2) We can also form

$$\frac{|\mathfrak{S}\mathfrak{C}|}{|E_{\rm crit}|} \ . \tag{121}$$

$$\frac{|\mathfrak{G}\text{raph}(N \,|\, E \ = \ E_0)}{|\mathfrak{G}\text{raph}(N)} \tag{122}$$

is a more general concept of this kind.

Part II
Topological Graph Theory

Definition 1 For w a vertex in the graph $\mathfrak{G}$, the vertex x is *adjacenct* to w if there is an edge e joining x and w. Let us write this as a relation $\sim_a$.

Remark 1 $\sim_a$ is symmetric:

$$x \sim_a w \;\Leftrightarrow\; w \sim_a x \; . \tag{123}$$

It is not however reflexive (no loops) nor transitive.

Definition 2 The set of vertices adjacent to a vertex $w \;\in\; \mathfrak{G}$ is the *neighbourhood* of w,

$$\mathfrak{N}(w) \;:=\; \{\, v \;\in\; \mathfrak{G} \,|\, v \sim_a \} \; . \tag{124}$$

malized q**Remark 2** We use this name to indicate that adjacency has some topological content.

Definition 3 For e an edge in the graph $\mathfrak{G}$, the edge f is *adjacent* to e if they share a common vertex v.

Remark 3 This can also be described as f, e both being incident to v.

Remark 4 This is another symmetric, non-reflexive, non-transitive relation.

Symmetry means that if $x \;\in\; \mathfrak{N}(w)$, then $w \;\in\; \mathfrak{N}(x)$.

Remark 5 Degree or valency can thus be rephrased as

$$d(w) \;=\; |\mathfrak{N}(w)| \;\; : \tag{125}$$

a notion of *local neighbourhood size*.

10 Independence, cover and domination

10.1 Vertex independence and cover

Definition 1 A vertex set $\mathfrak{I} \subseteq \mathfrak{V}(\mathfrak{G})$ is *independent*, alias *internally stable* [26] in $\mathfrak{G}$ if no pair v, w both in $\mathfrak{V}(\mathfrak{G})$ are adjacent.

Remark 1 This is most obviously a notion of separation: vertices separated by > 1 edge.

Definition 2 The *set of independent sets* supported by a graph $\mathfrak{G}$ is

$$\mathfrak{I}\text{ndep}(\mathfrak{G}) \ . \tag{126}$$

Definition 3 A set $\mathfrak{C} \subseteq \mathfrak{V}(\mathfrak{G})$ is a *vertex cover* [26] if every $e \in \mathfrak{E}(\mathfrak{G})$ is incident to at least one vertex of $\mathfrak{C}$.

Definition 4 The *set of vertex covers* supported by a graph $\mathfrak{G}$ is

$$\mathfrak{C}\text{over}(\mathfrak{G}) \ . \tag{127}$$

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Definition 5 An independent set $\mathfrak{I} \subseteq \mathfrak{G}$ is *maximum* if $\nexists$ an independent set $\mathfrak{I}'$ such that $|\mathfrak{I}'| \geq |\mathfrak{I}|$.

Definition 6 A vertex cover $\mathfrak{C} \subseteq \mathfrak{G}$ is *minimum* if $\nexists$ a vertex cover $\mathfrak{C}'$ such that $|\mathfrak{C}'| \leq |\mathfrak{C}|$.

Definition 7 The *independence number* alias *internal stability number*

$$\alpha(\mathfrak{G}) \ := \ \#(\text{vertices in a maximum independent set in } \mathfrak{G}) \ = \ \mathfrak{I} \ \in \ \overset{\max}{\mathfrak{I}\text{ndep}(\mathfrak{G})} \ |\mathfrak{I}| \ . \tag{128}$$

Definition 8 The *vertex covering number*

$$\beta(\mathfrak{G}) \ := \ \#(\text{vertices in a minimum vertex cover in } \mathfrak{G}) \ = \ \mathfrak{C} \ \in \ \overset{\max}{\mathfrak{C}\text{over}(\mathfrak{G})} \ |\mathfrak{C}| \ . \tag{129}$$

10.2 Vertex domination

Definition 1 A subset $\mathfrak{D}$ of vertices in a graph $\mathfrak{G}$ is a *vertex dominating set* alias *vertex external stability number* if every vertex not therein is adjacent to at least one which is.

Definition 2 The *set of vertex dominating sets* supported by a graph $\mathfrak{G}$ is

$$\mathfrak{D}\text{omina}(\mathfrak{G}) \ . \tag{130}$$

Definition 3 A dominating vertex set $\mathfrak{D}$ is *minimum* if $\nexists$ any dominating vertex set $\mathfrak{D}'$ such that $|\mathfrak{D}'| < |\mathfrak{D}|$.

Definition 4 The *vertex domination number* alias *vertex external stability number* is

$$\gamma(\mathfrak{G}) \ := \ \#(\text{vertices in a minimum dominating vertex set in } \mathfrak{G}) \ = \ \mathfrak{D} \in \overset{\min}{\mathfrak{D}\text{omina}(\mathfrak{G})} \ |\mathfrak{D}| \ . \tag{131}$$

10.3 Matchings and edge covers

Definition 1 A set $\mathfrak{M} \subseteq \mathfrak{V}(\mathfrak{G})$ is an *independent edge set*, alias *matching* in $\mathfrak{G}$ if no pair e, f both in $\mathfrak{M}$ are incident on a common vertex.

Definition 2 The *set of matchings* supported by a graph $\mathfrak{G}$ is

$$\mathfrak{M}\text{atch}(\mathfrak{G}) \ = \ \mathfrak{I}\text{ndep}_1(\mathfrak{G}) \ . \tag{132}$$

Definition 3 A set $\mathfrak{C}_1 \subseteq \mathfrak{V}(\mathfrak{G})$ is an *edge cover* if every $v \in \mathfrak{V}(\mathfrak{G})$ such that $d(v) > 0$ meets at least one edge of L.

Definition 4 The *set of edge covers* supported by a graph $\mathfrak{G}$ is

$$\mathfrak{C}\text{over}_1(\mathfrak{G}) \ . \tag{133}$$

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Definition 5 A matching $\mathfrak{M} \subseteq \mathfrak{G}$ is *maximum* if $\nexists$ a matching $\mathfrak{M}'$ such that $|\mathfrak{M}'| > |\mathfrak{M}|$.

Definition 6 An edge cover $\mathfrak{C}_1 \subseteq \mathfrak{G}$ is *minimum* if $\nexists$ an edge cover $\mathfrak{C}_1'$ such that $|\mathfrak{C}_1'| < |\mathfrak{C}_1|$.

Definition 7 The *edge independence number* alias *matching number*

$$\alpha_1 := \#(\text{edges in a maximum matching in } \mathfrak{G}) = \overset{\max}{\mathfrak{M} \in \mathfrak{M}\text{atch}(\mathfrak{G})} |\mathfrak{M}| \ . \tag{134}$$

Definition 8 The *edge covering number*

$$\beta_1 := \#(\text{ vertices in a minimum edge cover in } \mathfrak{G}) = \overset{\max}{\mathfrak{C}_1 \in \mathfrak{C}\text{over}_1(\mathfrak{G})} |\mathfrak{C}_1| \ . \tag{135}$$

10.4 Edge domination

Definition 1 A subset $\mathfrak{D}_1$ of edges in a graph $\mathfrak{G}$ is a *dominating edge set* alias *external stability edge set* if every edge not therein shares a vertex with an edge which is.

Definition 2 The *set of dominating edge sets* supported by a graph $\mathfrak{G}$ is

$$\mathfrak{D}\text{omina}_1(\mathfrak{G}) \ . \tag{136}$$

Definition 3 A dominating edge set is *minimum* if $\nexists$ any dominating edge set $\mathfrak{D}_1'$ such that $|\mathfrak{D}_1'| < |\mathfrak{D}_1|$.

Definition 4 The *edge domination number* alias *external stability edge number* is

$$\gamma(\mathfrak{G}) := \#(\text{edges in a minimum dominating edge set in } \mathfrak{G}) = \overset{min}{\mathfrak{D}_1 \in \mathfrak{D}\text{omina}_1(\mathfrak{G})} |\mathfrak{D}_1| \ . \tag{137}$$

10.5 Inter-relations

Proposition 1 [26, 23] Letting i run over 0-or-blank for vertices and 1 for edges, these obey the following.

i) the equality

$$\alpha_i + \beta_i = |\mathfrak{G}| \ . \tag{138}$$

ii) The inequality

$$\gamma_i \leq \alpha_i \ . \tag{139}$$

The 1-version of i) is known as *Gallai's Theorem*

10.6 Ratio version

Let $|\mathfrak{G}| \geq 1$.

Definition 1 The *vertex independence fraction* alias *internal stability fraction*

$$a := \frac{\alpha}{N} \ . \tag{140}$$

Definition 2 The *vertex covering fraction*

$$b := \frac{\beta}{N} \ . \tag{141}$$

Definition 3 The *vertex domination fraction* alias *external stability fraction*

$$c := \frac{\gamma}{N} \ . \tag{142}$$

Definition 4 The *edge independence fraction* alias *matching fraction*

$$a_1 \;:=\; \frac{\alpha_1}{N} \;. \tag{143}$$

Definition 5 The *edge covering fraction*

$$b_1 \;:=\; \frac{\beta_1}{N} \;. \tag{144}$$

Definition 6 The *edge covering fraction*

$$c_1 \;:=\; \frac{\gamma_1}{N} \;. \tag{145}$$

Proposition 2 Proposition 1 now simplifies to

i)

$$a_i \;+\; b_i \;=\; 1 \;. \tag{146}$$

ii)

$$\mathcal{C}_i \;\leq\; 1 \;. \tag{147}$$

The a_i, b_i and c_i are moreover probability-valued. With i) signifying that a_i and b_i are collectively-exhaustive and thus partial variables. The non-independent ratio

$$\mathcal{C}_i \;:=\; \frac{c_i}{a_i} \tag{148}$$

is the *external-to-internal stability ratio*.

Remark 1 We can thus rewrite

$$a \;=\; a_p \;=\; \frac{\alpha}{\alpha \;+\; \beta} \;. \tag{149}$$

$$b \;=\; b_p \;=\; \frac{\beta}{\alpha \;+\; \beta} \;. \tag{150}$$

11 Connectivity

Definition 1 A graph $\mathfrak{G}$ is *connected* [26] if there is a path between any two of its vertices, v and w.

Definition 2 A graph is *disconnected* if it is not connected.

Remark 1 Definition 1 is a path-connectedness concept, while Definition 2 is a definition by exclusion.

Definition 3 A graph is totally disconnected if no pair of its vertices are connected.

Remark 2 It thereby has no edges, and can also be thought about in simpler terms as a point cloud.

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Definition 4 A *connected component* of a graph is a maximally connected subgraph.

Remark 3 Graphs are partitioned into connected components. Graphs can be characterized by number of connected components and sizes of each component; this is far from necessarily a unique characterization.

Definition 5 A *cut vertex* alias *articulation vertex* is a vertex whose deletion would increase the number of components.

Definition 6 A *cut edge* alias *bridge* or *isthmus* is an edge whose deletion would increase the number of components.

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Definition 7 A set of vertices $\mathfrak{W} \in \mathfrak{V}(\mathfrak{C})$ is a *(vertex) separating set* if $\mathfrak{C} - \mathfrak{W}$ is disconnected.

Definition 8 A set of edges $\mathfrak{F} \in \mathfrak{E}(\mathfrak{C})$ is an *(edge) separating set* alias *disconnecting set* if $\mathfrak{C} - \mathfrak{F}$ is disconnected.

Definition 9 Let us denote the set of all vertex separating sets by

$$\mathfrak{S}\text{epar} . \tag{151}$$

Definition 10 Let us denote the set of all edge separating sets = disconnecting sets by

$$\mathfrak{D}\text{iscon} . \tag{152}$$

Definition 11 The (vertex) connectivity number of a graph $\mathfrak{G}$,

$$\kappa \ := \ \min_{\mathfrak{W} \in \mathfrak{S}\text{epar}(\mathfrak{G})} |\mathfrak{W}| . \tag{153}$$

Definition 12 The edge connectivity number of a graph $\mathfrak{G}$,

$$\kappa_1 \ := \ \min_{\mathfrak{F} \in \mathfrak{D}\text{iscon}(\mathfrak{G})} |\mathfrak{F}| . \tag{154}$$

Remark 4 Some texts denote this by λ. In our choice of notation, the 1 stands for the dimension of edges.

Proposition 1 (Whitney's inequality) For any graph $\mathfrak{G}$,

$$\kappa \ \leq \ \kappa_1 \ \leq \ \delta . \tag{155}$$

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Definition 13 A graph $\mathfrak{G}$ is *ν-connected* if

$$\kappa \ \geq \ \nu . \tag{156}$$

This is often referred to as k-connectivity: using k in the role of ν.

Remark 5 This is limited to the range

$$0 \ \leq \ \nu \ \leq \ \Delta_{\max}(\mathfrak{G}\text{raph}) \ = \ |\mathfrak{G}| \ - \ 1 . \tag{157}$$

Proposition 2 Excluding 0 from this range, ν-connectedness is guaranteed if for each vertex v,

$$d(v) \ \geq \ \frac{(N \ + \ \nu \ - \ 2)}{2} . \tag{158}$$

11.1 Connectivity ratios

Definition 1 The *connectivity per vertex*

$$k \ := \ \frac{\kappa}{N} \ . \tag{159}$$

Definition 2 The *edge connectivity per edge*

$$k_1 \ := \ \frac{\kappa_1}{E} \ . \tag{160}$$

Definition 3 The *edge-vertex connectivity ratio*

$$\mathcal{K} \ := \ \frac{\kappa_1}{\kappa} \ = \ \frac{k_1}{k\,\eta} \ . \tag{161}$$

Proposition 3 Whitney's inequality extends and simplifies to

$$0 \ \leq \ k \ \leq \ 1 \ , \tag{162}$$

$$0 \ \leq \ l \ \leq \ 1 \ . \tag{163}$$

Remark 1 This result motivates the following definition.

Definition 4 The *edge connectivity per minimal degree*

$$m \ := \ \frac{\kappa_1}{\delta} \ . \tag{164}$$

Remark 2 The l in the ratio form of Whitney's inequality is then the reciprocal of this,

$$l \ := \ \frac{1}{m} \ . \tag{165}$$

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Definition 5 The *extent-of-connectivity fraction* is

$$n \ := \ \frac{\nu}{N \ - \ 1} \tag{166}$$

Remark 3 This normalization entails a $\mathfrak{G}$raph rather than $\mathfrak{G}$ ratio.

Remark 4 Formulating a ratio version of the bound on ν-connectivity motivates the following definition.

Definition 6 The *n-connectivity adjusted degree fraction*

$$\widetilde{d} \ := \ \frac{2\,d}{|\mathfrak{G}| \ + \ n \ - \ 2} \ . \tag{167}$$

Proposition 4 The ratio form of the degree bound guaranteeing ν-connectivity is

$$0 \ \leq \ \widetilde{d} \ \leq \ 1 \ . \tag{168}$$

Remark 5 All of k, l and $\widetilde{d}$ are probability-valued.

12 Clique and intersection fractions

Definition 1 The *clique number* $\omega(\mathfrak{G})$ is the size of the largest clique (K_n subgraph) in a graph.

Definition 2 The *intersection number* $\iota(\mathfrak{G})$ is the minimum number of elements in a set $\mathfrak{X}$ such that $\mathfrak{G}$ is a graph representing the intersections of a family of subsets of $\mathfrak{X}$.

Remark 1 These are preented together here because intersection number is conceptually a clique *cover* number.

12.1 Ratio versions

Definition 1 The *clique fraction*

$$w \ := \ \frac{\omega}{N} \ . \tag{169}$$

Definition 2 The *intersection fraction*

$$i \ := \ \frac{\iota}{E} \tag{170}$$

Remark 1 This is probability-valued for $\mathfrak{G}$ connected.

13 Planar graphs

Definition 1 A *planar graph* is one that can be embedded in the plane without edges crossing.

Definition 2 Let F denote face number for a planar graph.

Proposition 1 (Edge-Face-girth inequality) [26].

$$2\,E \;\geq\; F\,g\;. \tag{171}$$

Remark 1 This suggests using the following ratio variable.

Definition 2 The *edges-per-face ratio variable*

$$\epsilon \;=\; \frac{E}{F}\;. \tag{172}$$

Then

$$2\epsilon \;\geq\; g\;. \tag{173}$$

Remark 2 This is a second example of comparing a ratio of graph numbers to a graph number, here the girth g.

13.1 Planar density

Remark 1 The theory of planar graphs involves the following key number of edges.

Definition 1 The *maximum planar edge number*

$$E_{\text{max-planar}} \;:=\; 3\,(\,N \;-\; 2\,) \tag{174}$$

Remark 2 This in turn supports the following ratio variable, which is conceptually a type of density.

Definition 2 The *planar edge density*

$$\rho_{\text{planar}} \;:=\; \frac{E}{E_{\text{max-planar}}} \;=\; \frac{E}{3\,(\,N \;-\; 2\,)}\;. \tag{175}$$

Remark 3 This is probability-valued provided that we are computing it for a planar graph.

Definition 3 The *first planar fraction* for a given N is

$$\mathcal{P} \;=\; \frac{6\,(N \;-\; 2)}{N(N-1)}\;. \tag{176}$$

13.2 Bipartite graph parallel

Remark 1 Bipartite graphs [26] exhibit the following parallel.

Definition 1 The *maximum bipartite edge number*

$$E_{\text{max-bipartite}} \;:=\; 2\,(\,N \;-\; 2\,)\;. \tag{177}$$

Remark 2 This in turn supports the following ratio variable, which is conceptually a type of density.

Definition 2 The *bipartite edge density*

$$\rho_{\text{bipartite}} \;:=\; \frac{E}{E_{\text{max-bipartite}}} \;=\; \frac{E}{2\,(\,N \;-\; 2\,)}\;. \tag{178}$$

Remark 3 This is probability-valued provided that we are computing it for a bipartite graph.

Definition 3 The *first bipartite fraction* for a given N is

$$\mathcal{B} \;:=\; \frac{4\,(\,N \;-\; 2\,)}{N\,(\,N \;-\; 1)}\;. \tag{179}$$

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Remark 4 Thus

$$\frac{E_{\text{max-planar}}}{E_{\text{max-bipartite}}} = \frac{\mathcal{P}(\mathfrak{G})}{\mathcal{B}(\mathfrak{G})} = \frac{3}{2} : \tag{180}$$

a fixed N-independent value.

Remark 5 We write explicit $\mathfrak{G}$ dependence here since Sec 9's second such are $\mathfrak{G}$raph entities, rather.

14 Thickness quantification of deviation from planarity

14.1 Standard presentation

Definition 1 The *thickness* [11, 12, 26] of a graph $\mathfrak{G}$ is

$$\theta = \min\#(\text{planar floors in a car park}) ; \tag{181}$$

more formally, the number of pairwise edge-disjoint spanning subgraphs in a decomposition of $\mathfrak{G}$.

Remark 1 This is 1 for planar graphs. Its minimal nontrivial value is 2 .

Proposition 1

$$\theta \geq \frac{E}{3(N - 2)} \tag{182}$$

is standard.

Proposition 2 For bipartite graphs,

$$\theta \geq \frac{E}{2(N - 2)} . \tag{183}$$

14.2 Rational presentation

Definition 1 The *normalized thickness* is the thickness per planar density,

$$\Theta := \frac{\theta}{\rho_{\text{planar}}} . \tag{184}$$

Definition 2 The *bipartite normalized thickness* is the thickness per bipartite density,

$$\Theta_{\text{b}} := \frac{\theta}{\rho_{\text{bipartite}}} . \tag{185}$$

Proposition 3

$$\Theta \geq 1 \tag{186}$$

Proposition 4 For bipartite graphs, (183) becomes

$$\Theta_{\text{b}} \geq 1 \tag{187}$$

Together with thickness being non-negative, both of the above are confined to lie between 0 and 1. It needs to be an infinite-sized graph, however, to drive our variables down to 0.

15 Fractional planarity at the level of $\mathfrak{G}$raph

Definition 1 The *second planarity fraction*

$$\mathcal{P}' := \frac{|\mathfrak{P}\text{lanar}(N)|}{|\mathfrak{G}\text{raph}(N)|} . \tag{188}$$

Definition 2 The *fractional n-planarity*

$$\pi_n(N) := \frac{|\mathfrak{P}\text{lanar}(N \,|\, E = n)|}{|\mathfrak{G}\text{raph}(N)|} \tag{189}$$

Definition 3 The *fractional t-planarity*

$$\pi_t(N) \ := \ \frac{|\mathfrak{P}\text{lanar}(N \,|\, \theta \ = \ t)|}{|\mathfrak{G}\text{raph}(N)|} \tag{190}$$

Remark 1 $\pi(N)$ can be expanded as a sum in either of the two previous such.

Remark 2 We can follow suit in each case starting with the following.

Definition 4 $|\mathfrak{C}\mathfrak{G}$raph(N)$|$

Definition 5 $|\mathfrak{C}\mathfrak{p}$lanar(N)$|$

Definition 6 The *connected fractional planarity*

$$\mathcal{CP}' \ := \ \frac{|\mathfrak{C}\mathfrak{p}\text{lanar}(N)|}{|\mathfrak{C}\mathfrak{G}\text{raph}(N)|} \ . \tag{191}$$

Definition 7 $|\mathfrak{B}$ipartite(N)$|$

Definition 8 The *bipartite fractional planarity*

$$\mathcal{B}' \ := \ \frac{|\mathfrak{B}\text{ipartite}(N)|}{|\mathfrak{G}\text{raph}(N)|} \ . \tag{192}$$

Remark 3 These compose in whichever order.

16 Graph colouring

16.1 Standard presentation

Definition 1 The *(vertex) chromatic number* [22, 26] χ of a graph $\mathfrak{G}$ is the minimal number of classes of $\mathfrak{V}(\mathfrak{G})$ such that no edge joins two vertices within the same class.

Definition 2 The *edge chromatic number* χ_1 of a graph $\mathfrak{G}$ is the minimal number of classes of $\mathfrak{E}(\mathfrak{G})$ such that no edges meeting the same vertex are within the same class.

Remark 1 The classes in question are meaningless-label colouring classes.

Proposition 1 (basic).

$$\chi \ \geq \ \omega \ . \tag{193}$$

Proposition 2 (basic) [26].

$$\chi \ \geq \ \frac{|\mathfrak{G}|}{\alpha} \tag{194}$$

Proposition 3 (Brooks' inequality) [6]

$$\chi \ \leq \ \Delta \ . \tag{195}$$

Proposition 4 (Vizing) [13, 26, 23].

$$\chi_1 \ = \ \Delta \ \text{ or } \ \Delta \ + \ 1 \ . \tag{196}$$

Definition 3 These two cases are called *Vizing classes* 1 and 2 respectively

16.2 Rational presentation

Definition 1 *Chromatic number per maximum clique*

$$x \ := \ \frac{\chi}{\omega} \ . \tag{197}$$

Remark 1 This is a ratio of two structural quantifiers: global over local.

Ratio form of Proposition 1 x is probability-valued.

Ratio form of Proposition 2

$$a \ \geq \ \frac{1}{\chi} \ . \tag{198}$$

Remark 3 a and χ come out compared upside down.

Definition 2 *Chromatic number per maximum degree*

$$\mathcal{X} \ := \ \frac{\chi}{\Delta} \ . \tag{199}$$

Remark 4 This is also a normalization by a local quantifier.

Ratio form of Proposition 3 (Brooks' inequality) $\mathcal{X}$ is probability-valued.

Definition 3 *Edge chromatic number per maximum degree*

$$x_1 \ := \ \frac{\chi}{\Delta} \ . \tag{200}$$

Remark 5 Then Vizing class 1 corresponds to

$$x_1 \ = \ 1 \ , \tag{201}$$

while Vizing class 2 correspondes to

$$x_1 \ = \ 1 \ + \ \Delta^{-1} \ . \tag{202}$$

which for finite graphs is covered by the inequality

$$x_1 \ > \ 1 \ . \tag{203}$$

16.3 Arenas

Definition 1 The *space of* χ-colourable graphs is

$$\mathfrak{C}\text{olour}(N, \chi) \ . \tag{204}$$

Definition 2 The *space of* χ_1*-edge-colourable graphs* is

$$\mathfrak{C}\text{olour}(N, \chi_1) \ . \tag{205}$$

Remark 1

$$\mathfrak{G}\text{raph}(N) \ = \ \sum_{\chi=0}^{\Delta} \mathfrak{C}\text{olour}(N, \chi) \tag{206}$$

and also

$$\mathfrak{G}\text{raph}(N) \ = \ \sum_{\chi_1=\Delta}^{\Delta\,+\,1} \mathfrak{C}\text{olour}(N, \chi_1) \ . \tag{207}$$

17 Conclusion

We have given an account of Graph Theory in terms of hitherto seldom used relative, ratio and and dilatational variables. This amounts to introducing rational graph functions in place of cardinal graph functions. In the process, we have staked a claim on systematically naming everything we have introduced.

We showed how these variables simplify a number of inter-relations and inequalities, in parallel to our simplification of Euclidean Geometry inequalities in [40, 41]. We gave some indication of limitations starting to appear with some of the more complicated inequalities.

The material covered is moreover a useful preliminary for a detailed discussion of notions of structure in graphs and networks: a modern and very highly applicable topic [29, 31].

Another frontier we leave to another paper [50] is the topology of [graphs]: graphs modulo complementation.
Studies of network structure quantifiers [29] − notions of centrality, of average path distance, of dissimilarity, of overcrowding, clustering coefficients... − also benefit from Relational or Shape-Theoretic reformulations.
Some other directions some readers might consider developing are as follows.

1) Extend the current article to digraphs, multigraphs, hypergraphs and complexes.

2) Consider weighted versions the notions of density, average, and so on, that we describe above.

Acknowledgments

To my wisest friend. I also thank participants at the Applied Combinatorics Discussion Group, and the Global and Combinatorial Methods for Fundamental Physics Summer School 2021 for discussions.

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