On the type of Projection involved in forming Dirac Brackets

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Abstract

We consider the sense in which Dirac brackets are projections of Poisson brackets. In the process, we define a Dirac tensor, a Dirac projector and a Dirac–Jacobi tensor.

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1 Introduction

The *Dirac bracket* (see [2, 4] and e.g. [5, 8, 12, 20, 24, 32] for reviews covering it) gives one of the means of [12, 32] dealing with *second-class constraints* [2, 5], which needs to be done prior to quantization. It can be viewed as *projecting out* the *irreducible* second-class constraints [5], which arise in the course of Dirac's Algorithm for constrained systems [2, 4, 5, 12, 24, 25, 28, 32]. In the current note we consider what kind of projection is involved. In the process, a Dirac tensor, a Dirac projector and a Dirac–Jacobi tensor.

Some further motivation is as follows. Forming the Dirac bracket admits chain and digraph generalizations [33]. The Dirac Algorithm itself has recently been shown to fruitfully generalize to the Generalized [26] Lie Algorithm [1]. Various Linear-Algebraic and especially Combinatorial innovations are being made in recent and concurrent reviews of the Dirac (or Generalized Lie) Algorithms [28, 32, 33]. Therein, these algorithms are moreover argued to be central to Background-Independent resolutions [24, 25, 26, 28] of the Problem of Time [13, 14, 19, 24].

2 A basic Euclidean/Linear Algebra model of projection

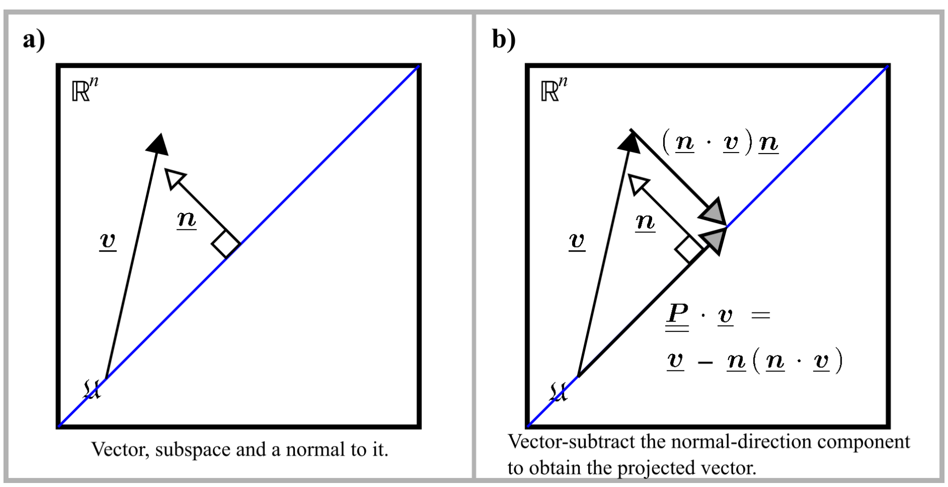

Figure 1:

Structure 1 *Projection* is often first encountered in the geometrical context of Fig 1 [18], giving the formula in components,

$$P_{ij}\, v_j \;=\; v_i \;-\; n_i\, n_j\, v_j \;. \tag{1}$$

Coordinate-free notation is however preferable for conceptual thinking. So in the present context,

$$\underline{\underline{\boldsymbol{P}}} \cdot \underline{\boldsymbol{v}} \;=\; \underline{\boldsymbol{v}} \;-\; \underline{\boldsymbol{n}}\,(\,\underline{\boldsymbol{n}} \cdot \underline{\boldsymbol{v}}\,) \;. \tag{2}$$

Here a vector $\boldsymbol{v}$ is being acted upon by a *projection operator* alias *projector* $\boldsymbol{P}$: a 2-tensor

$$\underline{\underline{\boldsymbol{P}}} \;=\; \underline{\underline{\mathbb{I}}} \;-\; \underline{\boldsymbol{n}}\,\underline{\boldsymbol{n}} \;. \tag{3}$$

This removes – i.e. projects out – the vector's component $(\,\underline{\boldsymbol{n}} \cdot \underline{\boldsymbol{v}}\,)$ in the direction of the unit normal $\boldsymbol{n}$ to the subspace $\mathfrak{U}$ that is being projected down to (Fig 3). The second term in (3) is the *dyad* formed from $\boldsymbol{n}$.

Remark 1 Two key properties of projectors are as follows.

$$\boldsymbol{P}^2 \;=\; \boldsymbol{P} \;\; \text{(idempotency)} \; . \tag{4}$$

$$\boldsymbol{P}^\dagger \;=\; \boldsymbol{P} \;\; \text{(self-adjointness)} \; . \tag{5}$$

By (5), $\boldsymbol{P}$'s eigenvalues must be real. A fortiori, by (4) and the Cayley–Hamilton Theorem, they must be 0 and 1. These correspond to yes/no answers to the question 'does it lie in $\mathfrak{U}$?'

Remark 2 In situations with multiple distinct projection operators, corresponding to multiple distinct subspaces being projected down to, (4) generalizes to

$$\boldsymbol{P}_{\mathfrak{U}} \boldsymbol{P}_{\mathfrak{W}} \;=\; \boldsymbol{P}_{\mathfrak{U} \,\cup\, \mathfrak{W}} \; . \tag{6}$$

If $\mathfrak{U} \,\cap\, \mathfrak{W} \;=\; 0$, we can furthermore write this union as a direct sum, $\;\oplus\;$.

Remark 3 For Quantum Mechanics (QM) [16] in Dirac's bra-ket notation,

$$P|\Psi\rangle \;=\; |\Psi\rangle \;-\; \left(\sum_i |\psi_i\rangle\langle\psi_i| \right) |\Psi\rangle \;=\; \left(\mathbb{I} \;-\; \sum_j |\psi_j\rangle\langle\psi_j| \right) |\Psi\rangle \; . \tag{7}$$

This is a Linear Algebra formulation, which extends moreover to cover the infinite-d (Hilbert space) case (relevant in both QM and Linear Methods). In coordinate-free notation, the projector here takes the form

$$\underline{\underline{\boldsymbol{P}}} \;=\; \underline{\underline{\mathbb{I}}} \;-\; |\,\_\,\rangle\langle\,\_\,| \; . \tag{8}$$

3 Further diversity of notions of projection

Generalization 1 Projection can be carried out for tensors in curved space. E.g. for a 2-tensor $\boldsymbol{S}$,

$$\overline{\overline{\underline{\underline{\mathbf{P}}}}}_{\boldsymbol{N}} \circ \overline{\overline{\boldsymbol{S}}} \;=\; \overline{\overline{\boldsymbol{S}}} \;-\; \overline{\overline{\boldsymbol{N}}}\,\underline{\underline{\boldsymbol{N}}} \circ \overline{\overline{\boldsymbol{S}}} \;=\; \left(\overline{\overline{\underline{\underline{\mathbb{I}}}}} \;-\; \overline{\overline{\boldsymbol{N}}}\,\underline{\underline{\boldsymbol{N}}} \right) \circ \overline{\overline{\boldsymbol{S}}} \; . \tag{9}$$

This involves a normal 2-tensor $\boldsymbol{N}$, corresponding to projecting out a subspace of the space of 2-tensors. Also, $\circ$ stands for double contraction.

Generalization 2 In curved (semi-)Riemannian geometries, a well-known formula is, in components,

$$h_{\mu\nu} \;=\; g_{\mu\nu} \;-\; \epsilon\, n_\mu n_\nu \; . \tag{10}$$

In coordinate-free form,

$$\underline{\underline{\boldsymbol{h}}} \;=\; \underline{\underline{\boldsymbol{g}}} \;-\; \epsilon\, \underline{\boldsymbol{n}}\,\underline{\boldsymbol{n}} \; . \tag{11}$$

The induced metric $\boldsymbol{h}$ on a hypersurface $\boldsymbol{\Sigma}$ is thus related to the bulk manifold $\mathfrak{M}$'s metric $\boldsymbol{g}$. Here the *signature* $\epsilon \;=\; 1$ in Riemannian Geometry, but $\epsilon \;=\; -1$ in e.g. GR spacetimes's semi-Riemannian Geometry [11]. Also greek indices are taken to run over 1 more value than latin ones.

Remark 1 1) and 2) furthermore compose. In Riemannian Geometry, one can moreover use (11) as a smallest projector block. E.g. for a (2, 0)-tensor in the Riemannian case, one can form

$$\overline{\overline{\underline{\underline{\boldsymbol{P}}}}}_{\boldsymbol{n}} \;:=\; (\,\underline{\underline{\boldsymbol{g}}} \;-\; \underline{\boldsymbol{n}}\,\underline{\boldsymbol{n}}\,) \circ \overline{\overline{\boldsymbol{S}}} \circ (\,\underline{\underline{\boldsymbol{g}}} \;-\; \underline{\boldsymbol{n}}\,\underline{\boldsymbol{n}}\,) \; . \tag{12}$$

Remark 2 So, while (9) maintains simple dyad form for projections of a 2-tensor, (12) involves a quartic including dyad of dyads.

Remark 3 Projections moreover have more general scope than Riemannian Geometry. For Affine, and indeed Projective, Geometry support such [21, 3]. And so do Symplectic, Contact and Poisson Geometry [15, 17, 23, 20].

4 The Dirac bracket as a projector

Structure 1 The current note's mathematical arena is *Poisson Geometry* and *Poisson Algebra* [15, 20]. Poisson brackets $\{\ \_\ ,\ \_\ \}$ can be reformulated in terms of the *Poisson tensor* $\boldsymbol{\pi}$ according to

$$\{ F, G \} \ = \ \underline{\boldsymbol{\partial}} F \cdot \overline{\overline{\boldsymbol{\pi}}} \cdot \underline{\boldsymbol{\partial}} G \ . \tag{13}$$

F and G are here arbitrary phase space functions. $\boldsymbol{\pi}$ is thus an antisymmetric bilinear tensor. It is presented as specifically a (2, 0)-tensor since

$$\boldsymbol{\pi} \ = \ \boldsymbol{\omega}^{-1} \ . \tag{14}$$

I.e. it is the inverse of the somewhat better-known *symplectic 2-form* [20] $\underline{\underline{\boldsymbol{\omega}}}$, whenever this exists. For Poisson spaces are more general than symplectic spaces, out of not having nondegeneracy requirements [20].

What a Poisson tensor is still required to have is a vanishing associated *Jacobi tensor* $\boldsymbol{J}$: a (3, 0) tensor as per Fig 2.b). Thereby, Poisson brackets close under Jacobi's identity, thus forming Lie algebras. Being derivations as well, Poisson brackets form a fortiori Poisson algebras.

Sructure 2 The *Dirac bracket* is usually written along the following lines.[1]

$$\{\ \_\ ,\ \_\ \} \ := \ \{\ \_\ ,\ \_\ \} \ - \ \{\ \_\ , \underline{\mathcal{I}} \} \cdot \{ \overline{\mathcal{I}}, \underline{\mathcal{I}}' \}^{-1} \cdot \{ \overline{\mathcal{I}}',\ \_\ \} \ . \tag{15}$$

The $\mathcal{I}$ are irreducible second-class constraints [5]. The primes keep clear which pairings of these are contracted.

Remark 1 To view the Dirac bracket as a projection of the Poisson (2, 0)-tensor $\boldsymbol{\pi}$. The $\mathcal{I}$ play here the role of projected-out normals. What one is seeking to construct is the projection By the Poisson tensor being a (2, 0) tensor, it is standard for this projection to be a quartic polynomial in its normals [c.f. (12)].

Structure 3 What is less standard is the lack of availability of a metric, alongside the availability of the Poisson tensor itself in a further underlying structural role. The outcome is that the Poisson tensor twists the four normal factors into a more complicated linear structure than (12)'s dyad of dyads. This gives Fig 2.c)'s first structure $\boldsymbol{\vartriangle}$, for which we coin the name *Dirac tensor*. This then factorizes to give this subfigure's second structure: the *Dirac projector* $\boldsymbol{DP}$. (If which $\mathcal{I}$ is involved were to be ambiguous, we would write $\boldsymbol{\vartriangle}_{\mathcal{I}}$ and $\boldsymbol{DP}_{\mathcal{I}}$). These are (2, 0) and (1, 1) tensors respectively. This follows from the Dirac bracket being some more reduced state space's Poisson bracket [7], by which the Dirac tensor is a species of Poisson tensor.

Structure 4 The Dirac tensor thus furthermore enjoys all the properties of the Poisson tensor. In particular, it is antisymmetric and its corresponding Jacobi tensor, for which we coin the name *Dirac–Jacobi tensor* and the notation $\boldsymbol{DJ}$ (or $\boldsymbol{DJ}_{\mathcal{I}}$), vanishes. This totally-antisymmetric (3, 0) tensor is detailed in Fig 2.d).

[1]Distinguishing state space variables (copper) from constraints (fireopal) is an extension [30, 31, 32, 33] of the usual presentation [5]. Also the Einstein sum convention is in use, followed by coordinate-free reformulation.

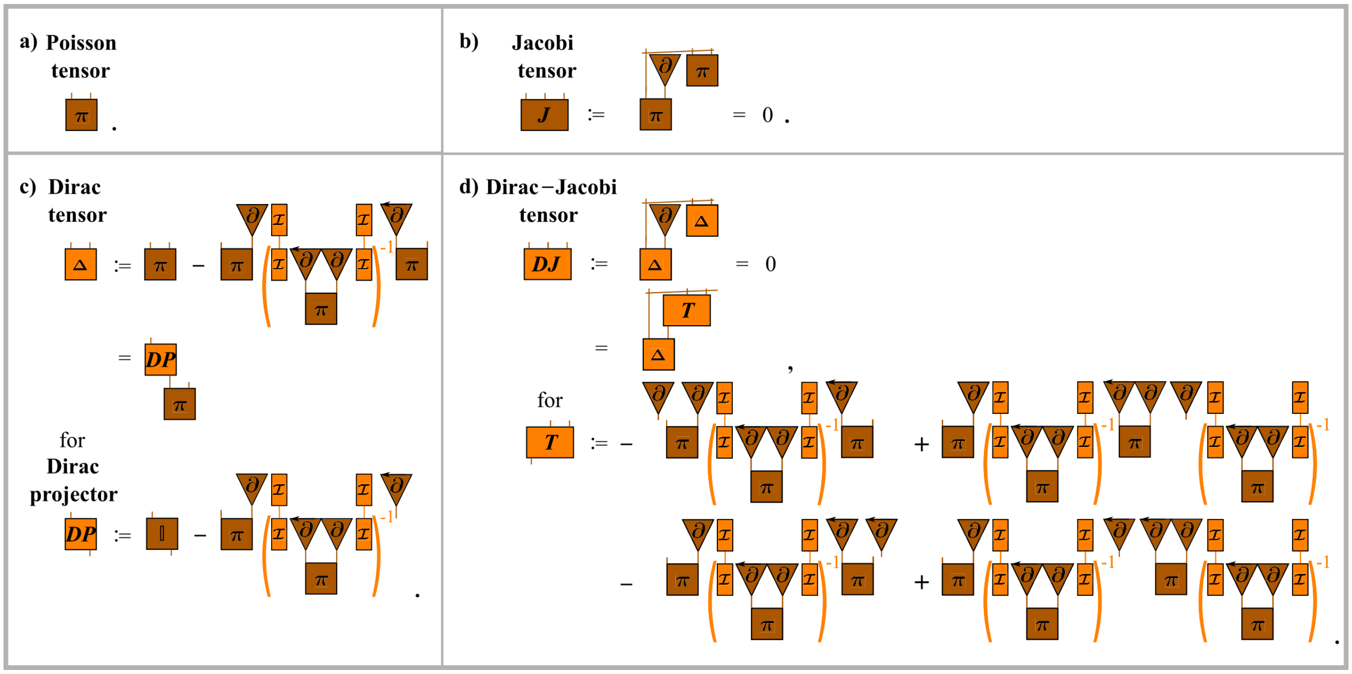

Figure 2: We present this note's key results as c) and d) of this figure. The language this is written in is the vertical variant of Penrose birdtracks [6], as further augmented to cover two kinds of tensor at once: copper state space indexed and fireopal irreducible second-class constraint indexed. The slash stands for antisymmetrization. See [32] for further details about, and scope of, such Rainbow Tensor Calculi.

5 Conclusion

We have explained the sense in which the Dirac bracket is a projective notion. This can be expressed as a (2, 0) *Dirac tensor* △, within which a *Dirac projector* (1, 1) tensor ***DP*** can be isolated. The Dirac tensor also has an associated (3, 0) Dirac–Jacobi tensor ***DJ***.

With the Dirac bracket being a projection, forming a such at each step down the chain of iterations of the Dirac Algorithm gives the same answer as forming it for the whole chain all at once by property (6). This is used in establishing a Theorem in [33].

The current note's constructions carry over to Lie Theory as well. Therein, the Lie–Dirac bracket gives the corresponding projecting out of second-class generators. We thus now know that this can be thought of as a (2, 0) Lie–Dirac tensor containing a (1, 1) Lie–Dirac notion of projector and closing in accord with a (3, 0) Lie–Dirac–Jacobi tensor.

Acknowledgments

I thank participants at the Global and Combinatorial Methods for Fundamental Physics Summer School 2021 for discussions.

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