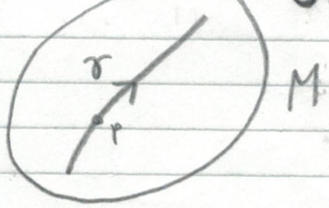
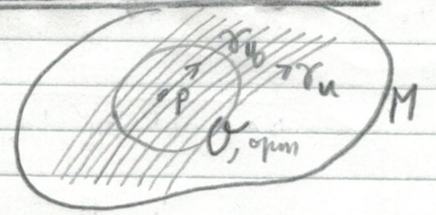
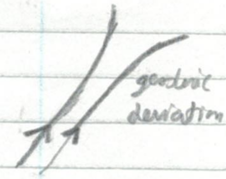
(p1) 5

§5 Second Variation of Arclength

§5.1 Introduction







1 geodesic involves Γ

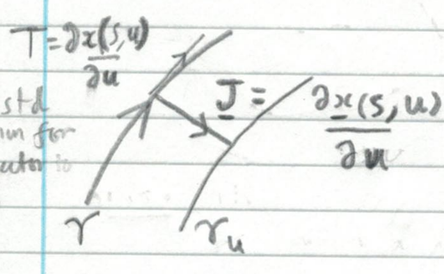
looking around a pt p locally involves multiple geodesics $\leftrightarrow$ curvature R

~ a non-intersecting family: a congruence
1-parameter u in the 2d picture.

A congruence in $\overset{\text{open}}{\mathcal{O}} \subset M$ is a family of curves
s.t. precisely one curve in this family passes though each point $q \in \mathcal{O}$.

There is a 1:1 correspondence between a congruence and the v.f. of its tangents in $\mathcal{O}$.
We consider this in §5.2.

Another characterization is in terms of a family of varied curves



~ the std expression for t vector to γ.

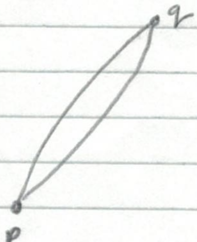
s the arclength along the geodesic.

γ_u is still restricted to itself be a geodesic.
(so it's a restricted family of varied curves)

We consider this in §5.3. It is a fast route to obtaining a d.e. for geodesic deviation.

We then approach this in a more general manner from a variational principle in §5.4–5.5
(2nd variation of arc, to geodesics arising from 1st variation)

One of the most interesting features is that of eventual re-intersection of two geodesics originating from a given point:



meridian great circles is a basic example

§5.6) The points p and q are then called conjugate. $\leftrightarrow$ Jacobi field vanishes at p, q.

The existence or otherwise of conjugate points turns out to quite often be useful in proofs by contradiction.

In particular, conjugate points are considered in this manner in {the Jacobi test for maxmin / proof of the GR Singularity Thms

We finally simplify the 2nd variation using an index presentation in §5.7, allowing us to tap in to some basic ODE theory. Curvature sign dependent results ensue from §5.5–7.